Solving the inverse problem of moments using the Christoffel-Darboux kernel

by

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Let μ be finite Borel measure having compact and infinite support $S = \operatorname{supp}(\mu)$ in the complex plane $\mathbb{C}$, and consider the Lebesgue space $L^2(\mu)$, with inner product $\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}d\mu(z)$ and norm $\|f\|_{L^2(\mu)} := \sqrt{\langle f, f\rangle_\mu}$.

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ denote the sequence of orthonormal polynomials associated with μ; that is, the unique sequence of the form

$$p_n(\mu, z) = \gamma_n(\mu)z^n + \cdots, \quad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \dots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot)\rangle_\mu = \delta_{m,n}$.

An extremal problem leads to the sequence $\{\lambda_n(\mu, z)\}_{n=0}^{\infty}$ of the so-called *Christoffel functions* associated with the measure μ. These are defined, for any $z \in \mathbb{C}$, by $\lambda_n(\mu, z) := \inf\{\|P\|^2_{L^2(\mu)}, P \in \mathbb{P}_n \text{ with } P(z) = 1\}$, where $\mathbb{P}_n$ stands for the space of complex polynomials of degree up to n. Using the Cauchy-Schwarz inequality it is easy to verify that

$$\frac{1}{\lambda_n(\mu, z)} = \sum_{k=0}^{n} |p_k(\mu, z)|^2, \quad z \in \mathbb{C}.$$

Hence, $\lambda_n(\mu, z)$ is the reciprocal of the diagonal of the *Christoffel-Darboux* kernel

$$K_n(\mu, z, \zeta) := \sum_{k=0}^{n} \overline{p_k(\mu, \zeta)}p_k(\mu, z).$$

The purpose of the talk is describe an reconstruction algorithm, based on the asymptotics of the Christoffel-Darboux kernel, for computing approximations to the support S of μ. The input of the algorithm is a finite set of the complex moments

$$\int z^m \overline{z}^n d\mu(z), \quad m.n = 0, 1, \dots,$$

of the measure μ. This leads to applications in geometric tomography and the detection of outliers and anomalies in statistical data.